A Development of Integer Arithmetic – Everything Comes from the Definition

R. West 6/27/14
rgwest@ucdavis.edu
CSU Sacramento Mathematics Project

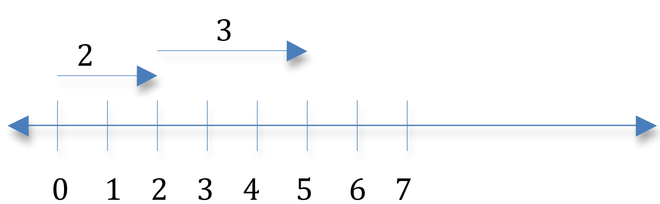
Introductory Notes:

- All skills will be developed from the definition of a negative number and symbolic manipulation will follow.
- The general pattern of instruction for each topic will be
  - Do the first example using what the students know about negative numbers.
  - Do several more examples until many of the students see the general pattern asking the students to make a prediction for the answer each time.
  - The students will see and articulate a "short cut" for doing the problem- The students explain in their language why this pattern always works.(A question to ask and a possible student explanation will be provided for each topic in the outline)
- Operations will mean the same thing when used with integers as with whole numbers – consistency with whole number arithmetic.

I. Review of Adding Whole Numbers and Fractions on the Number Line

A. Adding Whole Numbers

Draw a number line on the board, starting with 0, and ask the kids to draw a picture of 2 + 3. [This is intended to be review but if not, the following question should work anyway.
Ask, "Where should I start the 2? [at 0] "Tell me when to stop. [The kids should stop you when the arrow gets to 2.]
Ask, "How should I represent the 3? The goal is to get the following picture on the board.



Do enough of these so students are comfortable with putting the addition problem on the number line. It would be good to do some with fraction with common denominators.

The kids should be good at answering the questions; (assume the problem is 4 + 6 = 10.

2) Where does this arrow end?
3) Where, in relation to the 4 arrow, does the arrow representing 6 start?
4) Where does this arrow end?

Note: The notation $^-4$ will be used instead of -4 (note the raised position of the negative sign) to distinguish between the sign of the number and the symbol for the operation subtraction.

II. Locating Negative Integers on the Number Line

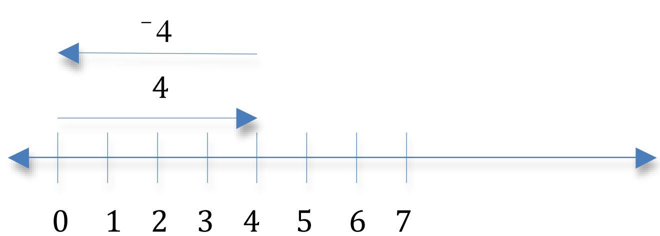
A. Using the Definition to Locate Negative Integers on the Number Line

Ask, "Suppose I told you negative 4 is the number I add to 4 to get 0. So $4 + {}^-4 = 0$. How could you represent this on a number line?

I would have them do this on their papers and check to see what they have. After checking, ask, (and record the answers as to create the picture shown below.

"What number did you start with?" (4)
"Where did the 4 start?" (0)
Where did it end?" (4)
"If we do this the same way as we did the whole number problems, where does the $^-4$ arrow need to start?" (at the end of the 4 arrow)
"Where did it end?" (0)

You should have the following on the board.



Now ask, "How many think addition is commutative?" [May need to ask something about order doesn't matter.]

Then, "What should $^-4 + 4$ equal?" Now ask them to draw the picture for $^-4 + 4$.

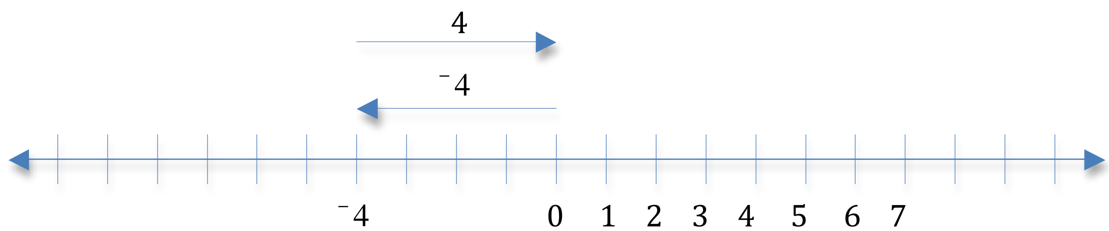
Note: We have not told the students how to read "$^-4$", it was just read while pointing to the numbers. **That is, ". . . draw the picture for 4 plus negative 4.**

Check papers to see what they have and then get the picture on the board asking the same questions as above in $4 + {}^-4 = 0$.

After getting the picture on the board ask, "So show me on your fingers how many places there are on the number line where you could add four and get back to 0?"

Some of the kids will likely have $^{-}4$ placed on the number line but if not, put a hash mark there and ask what number should go here? [If you don't get an answer, go back to a whole number addition problem and ask what number went at the end of the first arrow.]

> Note: I call the "if they don't get it" question above a "back pocket" question. That is, a question you can ask if the students if they don't get the first, big question. The idea is an attempt at avoiding over scaffolding and seeing if the students can use logic to come up with the answer more on their own.



Ask, "What is the same about the arrows?" (the length)
Ask, "What is different about the arrows?" (the direction)

> Note: These questions should be asked again after some other examples so the students see that this is always true for "a" and "^{-}a".

Ask, " If this is how we found $^{-}4$, find $^{-}5$ on your papers by drawing a picture." Check that they have the $^{-}5$ placed correctly on their number line. **You may need to preface this with "Given what we know about negative 4, what do you think is true about negative 5?"**

After adding negative 5 to the number line, place the rest of the hash marks and point to the hash mark corresponding to $^{-}3$ and ask, "What number should go here?"
Then ask, "Who can explain how we can be sure this is $^{-}3$ using the definition of $^{-}3$?"

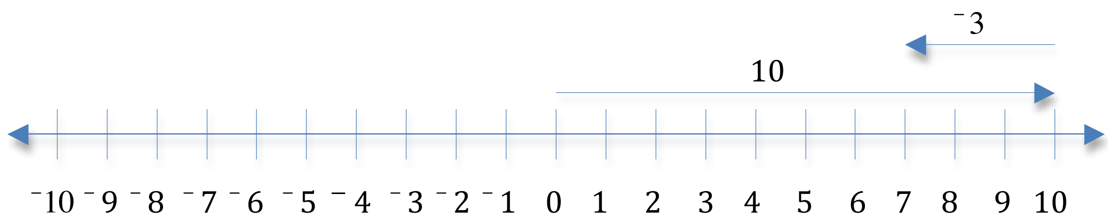
Desired response: "Since negative 3 + 3 equals 0, negative 3 has to be 3 units to the left of 0 so that when you add 3 you get back to 0."

III. Adding Integers on the Number Line

A. The sum of a positive and a negative with a positive sum (review adding two positive integers **on the number line** before doing this problem).

Ask, "How many can imagine a number line picture representing $10 + {}^{-}3$? "
Ask, "How many think the answer is going to be positive?"
"How many think negative?"
Then ask for conjectures on what the answer is will be.

Now have them draw the picture on their papers. At least some should have the following picture on their papers. Get this picture on the board and have them explain their reasoning.



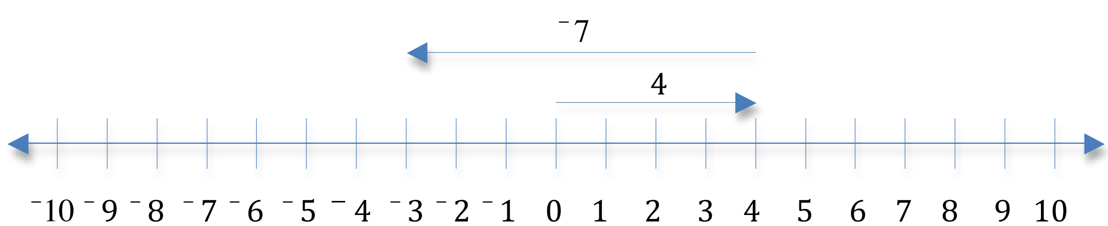
Repeat the same thing for $5 + {}^{-}2$ and other similar examples, asking for conjectures each time. Asking the students for conjectures before drawing the picture is intended to get them to look for the "short cut."

> Note: Some of the students are going to see that they can get the answer by subtracting. This is fine but it might be nice to wait until the end of the following section to let them tell the short cut. Once they see the short cut, it is important to have them go back to the number line and explain why the short cut works.

B. The sum of a positive and a negative with a negative sum.

Ask, "How many can imagine a number line representing $4 + {}^{-}7$? "
Ask, "How many think the answer is going to be positive?" "How many think negative?"
Then ask for conjectures on what the answer is will be.

Now have them draw the picture on their papers. At least some should have the following picture on their papers. Get this picture on the board and have them explain their reasoning.



Do several more examples, suggestions listed below, with some answers positive, some negative and some with the negative number first in the problem.

$5 + {}^{-}4$

${}^{-}6 + 3$

$2 + {}^{-}7$

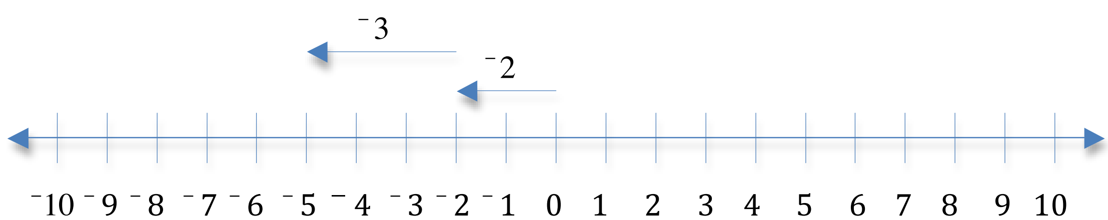
${}^{-}4 + 9$

When going over these problems, a bit of practice, ask the students to draw the picture for some of the early examples, then have them just imagine the picture and put down their answer. They can then progress to just using a "short cut" for the rest but every now and then make them come back to the picture to justify the short cut.

C. The sum of two negative integers.

Ask, "How many can imagine a number line representing $^{-}2 + {}^{-}3$? "
Ask, "How many think the answer is going to be positive?" "How many think negative?"
Then ask for conjectures on what the answer is will be.

Now have them draw the picture on their papers. At least some should have the following picture on their papers. Get this picture on the board and have them explain their reasoning.



Do more examples having the students tell what they think the answer is going to be and then drawing the picture. Once the students see you can just add . . .

Ask, "Can someone use the number line to explain why you can just add ignoring the negatives and then just make your answer negative?"

Desired response: "We are just adding a length of 2 to the left to a length of 3 to the left, so we have a length of 5 to the left."

IV. Subtracting Integers

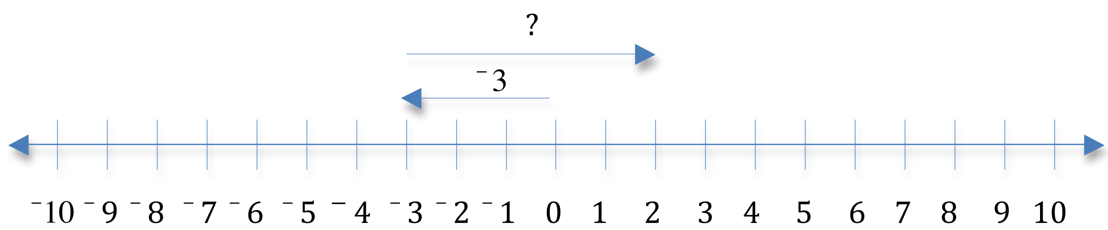
A. Subtracting a negative integer from a positive integer.

Ask, " How could I rewrite the equation 7 – 4 = ? as an addition problem?" [We want the equation, 4 + ? = 7

Do several more of these until the students clearly have the pattern.

Now ask, " How could we rewrite $2 - {}^{-}3$ = ? as an addition problem?" [We want $^{-}3 + ?$ = 2.]

Ask the students to imagine this on a number line and take conjectures for their answers then ask them to draw the picture on their papers. The hope is that they will have the following on their papers.



If the students have trouble with this, you can ask the following:
"Where should the $^-3$ arrow start?"
"Where should it end?"
"Where should the ? arrow start?"
"Where should it end?"
"What number is represented by the ? arrow?"

Many of the students will likely have the answer at this point but you can ask,
"Is the answer positive or negative?'
"Who can explain why?"

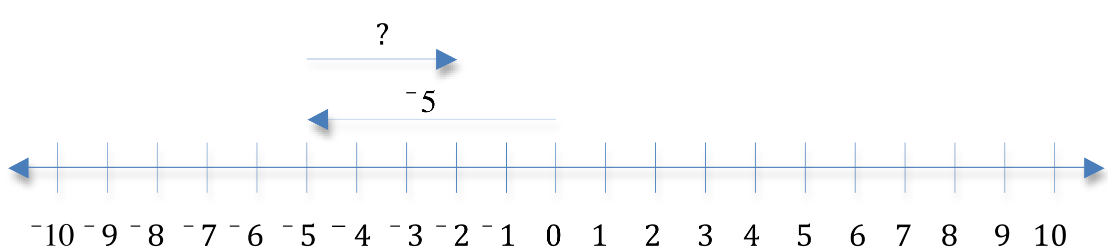
"How many think you know the answer?"
"Show me the answer on your fingers."

Do several more of these examples until the students are seeing that you can just, in the case of $2 - {}^-3$ is the same as 2 + 3. Once they see this, ask," can someone use the picture to explain why this works ?"

Desired student response: " If you split the ? arrow into two parts, it take 3 to get back to 0 and then 2 more to get to 2." Do a few more of these to make sure the students see the adding to get back to zero idea.

B. Subtracting a negative integer from a negative integer.

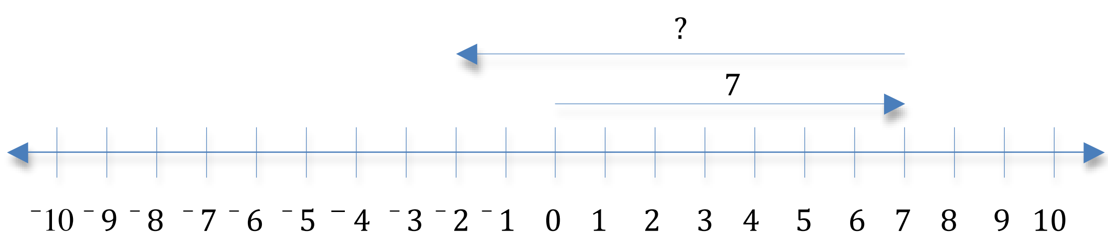
Ask, "What if I asked you to do $^-2 - {}^-5$?"
"How could we rewrite that as an addition problem?"
Get $^-5 + ? = {}^-2$
Then ask, "How many think the answer would be positive."
"How many think it will be negative?"
Ask for conjectures on the numeric value. Then ask them to draw it on a number line and see what they get. They should have the following:



It is likely that what to add to negative 5 to get negative 2 will be pretty obvious from the picture, both direction and length. After doing a few of these go back to $2 - {}^{-}3 = ?$ and review how you got to 0 first . . . Have them break the last problem down into two steps. That is to get from negative 5 to negative 2 you first add 5 to get to zero and then add negative 2 to get to negative two. Now ask, "How many people think it still looks like subtracting a negative is the same as adding the positive?"

C. Subtracting a positive integer from a negative integer.

Ask, "What do you think we will get for ${}^{-}2 - 7$?"
Take conjectures and then have them rewrite it as an addition problem and use the
At least some of them should have the following on their papers:



If a significant number of the students have the picture you can ask,
"Who can tall me the addition sentence?" and then,
"Who can explain how the number line picture tells what the answer must be?"

If the students don't have it done correctly, ask

"Where should the 7 arrow start?"
"Where should it end?"
"Where should the ? arrow start?"
"Where should it end?"
"What number is represented by the ? arrow?"

NEED GENERALIZTION ON EACH AND DESIRED RESPONSE . . .
\ *EXPLAIN HOW THEY ARE GOING TO GENERALIZE*

V. Multiplying Integers

A. Multiplying a positive by a negative

1. Multiplication is repeated addition.

Ask, "How could we rewrite $3 \cdot 4$ as an addition problem?"
[It doesn't matter which way they do this but lets assume $4 + 4 + 4$]

Now, "What do you think we would get for $3 \cdot {}^{-}4$?"
Take some conjectures and ask them to rewrite it as an addition problem on their papers to check their answers.

They should have ${}^{-}4 + {}^{-}4 + {}^{-}4$

Do several more of these until they are getting the answer without rewriting the problem as an addition problem. **Now ask, "why does** $7 \cdot {}^{-}4$ **= the negative of 7 times 4?"**

Desired response: "The problem tells you to take 4 steps to the left 7 times. So you are taking 7 times 4 steps all in the same direction."

2. Using the distributive law.

Review the distributive law so that they are comfortable writing, for example $(2 \cdot 3) + (2 \cdot 4) = 2 \cdot 7$. [When doing this it may be useful to write the 7 as 3 + 4 the first few times.]

Now, "Let's go back and look at $3 \cdot {}^{-}4$ another way."

"Who can use the distributive law to re-write $(3 \cdot 4) + (3 \cdot {}^{-}4) =$ "

Once you have $(3 \cdot 4) + (3 \cdot {}^{-}4) = 3 \cdot 0$ on the board, ask for the numeric values of $3 \cdot 4$ and $3 \cdot 0$ getting the following on the board:

$$\begin{array}{ccc} (3 \cdot 4) + (3 \cdot {}^{-}4) & = & 3 \cdot 0 \\ 12 \; + \quad ? & = & 0 \end{array}$$

and ask what $3 \cdot {}^{-}4$ must equal to make the sentence true.

NEED A WORD ABOUT COMMUTIVITY SO THE REVERSE IS TRUE

B. Multiplying a negative by a negative

Ask, "Who has an idea what we would get for ${}^{-}3 \cdot {}^{-}4$?"
After taking conjectures, ask, "Can we use repeated addition in this case?" [Hoping they will say no but if not you can ask if they can write down negative 3 of something.]

"When we did a positive times a negative we had a way other than repeated addition to do this. Does anyone remember the law we used?"

At this point you can either have them try to write distributive law sentence stressing the idea that you can solve equations that have one unknown or you can simply give them the sentence

$\left({}^-3\cdot{}^-4\right)+\left({}^-3\cdot 4\right)=$ and ask them what this would equal using distributive law. Get $\left({}^-3\cdot{}^-4\right)+\left({}^-3\cdot 4\right)=3\cdot 0$ on the board, ask them what the last two products equal so you have

$$\begin{array}{ccccc}\left({}^-3\cdot{}^-4\right) & + & \left({}^-3\cdot 4\right) & = & 3\cdot 0\\ ? & + & {}^-12 & = & 0\end{array}$$

on the board and ask what ${}^-3\cdot{}^-4$ must equal to make the sentence true.

VI. Division of Integers

USE DIVISION AS INVERSE OF MULT.