A Development of Fractions – Everything Comes from the Definition
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Introductory Notes:

- All skills will be developed from the definition of fraction and symbolic manipulation will follow.
- The general pattern of instruction for each topic will be

  - Do the first example using what the students know about fractions.
  - Do several more examples until many of the students see the general pattern asking the students to make a prediction for the answer each time.
  - The students see will see and articulate a "short cut" for doing the problem
  - The students explain in their language why this pattern always works.

  (A question to ask and a possible student explanation will be provided for each topic in the outline)

- Operations will mean the same thing when used with fractions as with whole numbers – consistency with whole number arithmetic.
- This approach is intended to develop the understanding of fractions as stressed by the Common Core Standards.

This document is intended to be useful for the first instruction starting in grade 3, for a review of a topic or for algebra teachers as well. For those teaching algebra and/or reviewing fractions, the recommendation is to start with the definition and move through the sequence below at a pace that works for the students. That is, avoid going first to the algorithm or symbolic explanation. [If that worked for them you wouldn't be having the problem in the first place.] The algebra sections are indicated with an asterisk after the standard outline numbering. The algebra sections are a generalization of the arithmetic.

I. Developing the Definition of Fraction.

A. Start with a line segment divided into 5 equal parts and shade two of them.
[Skip to II if the students can name the fraction.]

Ask, "How many pieces are in the whole?"
[Write the number in the denominator when they give it to you.]
Ask, " How many pieces are shaded?" [Write the number in the numerator when they give it to you.]
See note below on vocabulary.

Do the same thing with 5/7. [Try to make the line segment the same length as in the first example – leave the first example on the board.

Repeat the process with 5/6.

Do several more examples if needed until most have the idea. Remember to do paper problems early to check to see which students are catching on. [It would be good, but not critical, if everyone gets this at this point. We are going to start every new idea with an example of the above so there will be more opportunities for students to catch on.]

Note on Vocabulary – on the first example, I just asked how many pieces, . . . On the second example, I put the fraction segment on the board with hash-marks separating each equal section, pointed at the bottom and asked for a quiet hand (Q.H.) to tell me the denominator. Without pointing to the top, I then asked for the numerator. On the third one, I used the same order, did not point but used the vocabulary of denominator and numerator when asking the questions. This progression is meant to get the students to see the definition of each term in context rather than being told a definition.

On the second example, after cutting the segment into pieces, I asked them what was important about the size of the pieces. I got that they should be the same size. This set them up for the third one on which I made one piece significantly bigger than the others – they expressed their disapproval.

Note: While the primary model for fractions in this document will be on the number line it is important to use other models as well once the students have the idea. Also, when using line segments as the model, make sure to shade non-contiguous segments on the line segments to demonstrate that a fraction of the segment is shaded. That is, if three of five segments are shaded, it still represents 3/5.

B.* Algebra teachers could ask, "Suppose I have the fraction a/b. What does the "b" tell us? What does the "a" tell us?"

C. Ask, "How many people think you could draw the picture if I gave you a fraction?" "Okay, try $\frac{3}{4}$." [Have them do this on their papers.] After checking papers, ask, "How many pieces did you cut the whole into?"
Then, "And how many of those pieces are shaded?" In each case drawing what they say on the board.

II. Fractions on the Number Line. [Including fractions > 1]

A. Transforming the line segment into the number line from 0 to 1.
Draw a line segment cut into 5 equal pieces. Hold your fingers around the first piece, one finger on the left side of the segment the other on the first hash mark, and ask what fraction this represents. Write 1/5 under the appropriate hash mark. Repeat this for each hash mark (e.g. one finger on the left end, the other on the second hash mark outlining each distance with your fingers) getting 1/5, 2/5, 3/5 and 4/5 in the appropriate places underneath the hash-marks. Now ask, pointing at the right end, how many fifths would this be? Do the same for the left side of the segment. [Getting $\frac{5}{5}$ and $\frac{0}{5}$ on the picture.]

B. Extending the unit line segment to the number line.

Assuming you have written the fractions below the number line, ask what number 0/5 is the same as what number. Write 0 above that hash mark. Now ask, 5/5 would be the same as what number. Write 1 above that. Now you can ask, how many people think this is starting to look like a number line? Extend it out away, putting hash marks where the whole numbers would go and ask them for those values indicating that the distance between 0 and 1 is the same as the distance between 1 and 2 . . . [Note: always put the fractions on the same side of the number line (top-side or bottom-side), whole numbers on the other.]

C. Whole numbers as fractions

1. Now cut the new segments into fifths and have them start counting from the left with 0/5. Keep going until you get to at least 15/5. **[I have been doing this by having the students shout the fraction if it is under a whole number and whisper the fraction if not. I think this starts them focusing on which ones are whole numbers. It is at least fun.]**

Note: At this point is fine to skip to section III (equivalent fractions) and come back to the rest of II later.

2. Extend one of you number lines, for example the one in IIC where you stopped at 15/5. Extend it out to 8, with only the whole numbers labeled. Now ask, while pointing to the 8 on the number line, "How many fifths would this be?"

3. Do several more examples of this with different denominators until some of the students see a short cut**.** [They may count at first, then "skip count" before they actually see they can multiply. If they are counting, you can lead them to skip counting by asking them to chorally "count by fives" which should lead them to multiplication.]

Once the students see that they just multiply to get the answer, (use the 8 from E1 above) and ask, "Can someone explain to me why you would just multiply 8 by 5 to get the answer**?" [An alternative is to say, "8 times 5 could mean 8 groups of 5 or 5 groups of 8. In the context of the number line, which one is it?"]**

Desired student answer, " there are 8 groups of 5 on the number line so that is 8 times five pieces."

Optional: Get a picture on the board similar to those above with the number line cut into fourths, whole numbers from 1 to 7 and 1 through 4 also written as fractions.

Ask, " Who can give me an addition problem using the numbers on the number line whose sum is 7?" Say you get 3 + 4, ask, "Look at those fractions, how many think addition works with the fractions?" The idea is to get them to see that $\frac{12}{4}+\frac{16}{4}=\frac{28}{4}$**. Now you can ask for other addition problems with a sum of 7 telling**

the students they can use more than two numbers. It would be nice if they got to 1+1+1+ . . . +1 = 7 using fractions.

This is intended to expose the students to adding fractions but not meant to teach them adding fractions.

4. Do the same thing backwards. [Fractions to whole numbers.]
Draw a number line labeling the fractions from $\frac{0}{3}$ to $\frac{15}{3}$. Pointing to the $\frac{15}{3}$, ask, "What whole number would this be?" Then ask, "Can you turn any of the other fractions on the number line into whole numbers?"

Again, do this until some students see that they can do this with division.

Then ask, "Can someone tell me why division tells us the whole number?"

Desired students response, "The whole numbers are telling us how many groups of 3 there are on the line, and 15 divided by 3 tells us the same thing."

5* Do the same as in 1 but ask what fraction would be equivalent to 8 if the denominator is n? How about 2n? Also, "How could you express the whole number "a" as a fraction with denominator 5?"

D. Mixed Numbers

1. Draw a number line only indicating the whole numbers. Cut each pieces into say fourths, go out to the hash mark that corresponds to 5 ¾, and ask, "This would be 5 and how many fourths?" Write the numeral in the appropriate place and then ask, "How many fourths would this be?" The expectation is that they will use the skill they acquired in IIE1 to get the whole number converted and then add on the rest. Again, do several more examples until the students see that the general way to do this.

Now ask, "Who can explain why you just do (in this case) 5 times 4 and then add 3?" [It may be necessary to split this into two questions. For example "Who can explain where the 5 came from?" and "Who can explain where the 3 came from?"]

Desired students response, " We already know that up to 5 you have 5 groups of 4 so that is 20 and there are 3 more after the 5."

Note: The question here is "how many fourths would be in 5 ¾? It is not yet the standard question, "Convert the mixed number to an improper fraction?" The goal is to have them recognize that they can solve this by answering the whole number part first and add on the rest, exactly what is done in the traditional method. Ask the question in the traditional manner after they get proficient with the number line version.

5. So the same thing backwards. For example, ask (assuming the convention is still fraction below the number line and whole numbers about) "What number would go above $\frac{17}{3}$ on the number line?" Do several more examples of this until they see that you can do this with division and remainders

Ask, " Who can explain why the division problem tells us the answer on the number line?"

Desired student response, "When we divide we get 5 groups of 3, this is the 5 on the number line, with 2 left over that go to the right of 5."

6.* Ask, "Suppose I tell you a/b is to the right of 1, what can you say about the relationship between a & b? How about if a/b is to the left of one?

"Is $\frac{a+3}{a+1}$ less than 1?"

"What can you say about "x" if $\frac{2x+1}{9}$ < 1?"

III. Equivalent Fractions. (and simplifying fractions)

A. Get about 3 number line segment pictures of fractions on the board and have the students name the fraction representing the shaded part (using the same color to shade in each case). [Leave enough space under each picture for another picture – this would be good to do as a warm-up or "Daily Two."] Under the first example draw an exact copy of the line segment above and ask, "Suppose I cut each piece into two equal pieces (doing this as you say it) and I shade the same length as above. Starting at the left (make sure you are using a different color than above, ask, "Say STOP when I have shaded the same length." Now ask, "How many pieces do I have in the whole now?" and "How many pieces are shaded?" "What fraction does this represent?

Do this for all three pictures but dividing the pieces into a different number of pieces each time. [Note we are not talking about the fractions being equal yet.]

Now go back to the first picture and the one below it and ask, "Which is longer, the blue shaded part or the red shaded part?" (assuming those are the colors you used) Write the following three statements on the board

Red Length < Blue Length   Red Length = Blue Length   Red Length > Blue Length

and ask which is true. Now ask what is the fraction representing the red length and write the answer under that part of the equation and repeat the question for the blue part. At the end you should have (for example)

Red Length = Blue Length
3/5 = 6/10

Repeat this for the other examples on the board.

B. Generalize – getting kids to see the shortcut.
After doing enough examples so that the students are getting the idea with picture, do one without the picture. "Suppose I had the fraction 4/7 and I cut each piece into 2 pieces, what equivalent fraction would result?" Do this with several more fractions and cutting each new fraction into different amounts of pieces. After a several students seem to be getting it, ask, " Who can tell me how they are getting this answer?" The idea is to get them to see, and verbalize that they are multiplying the numerator and denominator by the same number.

Now, "so, what is the missing numerator in the equation $\frac{5}{8} = \frac{?}{24}$? Again, do a few of these to make sure most students have it.

Ask, "Can someone explain how you know the pictures on the number line would represent the same fractions?"

Desired response, "If you cut each piece into 3 pieces you have 3 times as many pieces."

It would be fun for the students at this point to be asked to come up with as many fractions as they can that are equivalent to say, 2/3.

C.* Start equivalence using numbers as above then ask, "Suppose I took my picture of 3/5 and cut each piece into "a" equal parts. What is the denominator of the resulting fraction? How about the numerator?" Do this with expressions other than a. [e.g.: x^2, x + 1, . . .]

D. Simplifying: Ask, " If I started with a fraction, cut the pieces into smaller pieces and ended up with 6/9, "What was my original fraction?" Then ask, "How many pieces did I cut each piece into?"

After doing several of these, pose another problem (e.g.: I ended up with 9/21, what did I start with?) and ask, "I don't want you to tell me my original fraction but what are you looking for that helps you find that fraction?" The goal is to get them to recognize that they are looking for a common factor, one that divides both the numerator and the denominator.

Give them a fraction like 42/60, (with a common factor 6 but don't tell them that), and ask, "Does anyone see any common factors?" If you get 2 or 3, reduce by that factor and ask the same question again.

Note: I would avoid using the word "simplify" when simplifying by the first factor. For the second factor (say 3) ask, "How many people can tell me what we would get if we simplified by a factor of 3?" The idea is to get them to learn the vocabulary in context.

I would do this both ways if you get both factors. First reducing by a factor of 2 then 3 and the reverse to see that it yields the same result. Had you gotten 6 as well, do the same thing. If you don't get 6, after reducing as above, ask, " Does anyone see a common factor that would have done the same thing in one step?"

Do several examples of the above. If greatest common factor is in the curriculum it should be talked about here. Divisibility tests are a possibility as well.

E.* After reducing a few numerical examples establishing that you are dividing the numerator and denominator by a common factor. Do some rational expressions. "How many people see a common factor in 7x/8x?" Gradually make the common factor more complicated.

Note: We are deliberately leaving out multiplying by 1 at this point. First, we have not discussed multiplication yet and secondly, this will give us a potentially powerful demonstration of the logic and consistency in mathematics when we discuss multiplication.

F. Families of Fractions: Write a fraction, leaving the numerator blank, with a 2 in the denominator. Ask, "What does the 2 tell us?" Then, "Suppose I cut each of the pieces into two equal pieces, what would the denominator be now?" Write

$\frac{\ }{2}, \frac{\ }{4}$ on the board. Then ask, "What would the denominator be if I cut the halves into 3 equal pieces?" Write $\frac{\ }{2}, \frac{\ }{4}, \frac{\ }{6},$ on the board and then ask, "What other denominators can we get if we start with a two in the denominator?" Now repeat the same questions starting with a denominator of 3, then 5 and possibly a few other prime numbers. These lists might be something one would want to post somewhere in the room.

Take one of the families and put a 1 in the first numerator and ask the students to fill in the rest of the numerators. This is a great place to review what they know about equivalent fractions as well as giving those who didn't get it the first time another chance.

Cover up the lists you already have, assuming they are posted somewhere, and ask, " I have a fraction with an 8 in the denominator, what family did it come from?" Do the same with say 9. Now ask the same questions for 6 and see if they see that it could have come from more than one family. Do this with a few more composite numbers.

Ask, "What number is in the thirds family and in the fifths family?" Do this with several more examples (no common factors) until the students recognize that the product is always in both families.

IV. Comparing Fractions. [Note: See Appendix A on comparing lengths with different units. Do some of these before comparing fractions.]

A. Ask, "Which is larger $\frac{1}{3}$ or $\frac{2}{3}$?" Do this with a few more examples with common denominators. Each time ask, "Are the pieces the same size" and " Which one has more pieces"

Ask, “So if I give you two fractions with the same denominator, how can you tell which one is larger?”

Desired response, “The pieces are the same size so you just look at the numerator to see which one has more pieces.”

B. Now switch to common numerators. For example, “Which is larger $\frac{2}{3}$ or $\frac{2}{5}$?” Do this with a few examples the goal being to get the students to explain that while the number of pieces is the same, the one with the smaller denominator has bigger pieces. [If I was doing this, on the first example I would tease the kids by saying they must be equal because both represent the same number of pieces.

Ask, “So if I give you two fractions with the same numerator, how do you tell which one is larger?”

Desired response, “Since they both have the same number of pieces, you just look to see which one has the larger pieces.”

C. One denominator is a factor of the other.
[It would be good to do a warm up problem comparing for example 8 feet and 3 yards as in appendix A. Also, a warm up problem should be converting say $\frac{3}{4}$ into a fraction with 8 in the denominator.]

Ask, “which is larger $\frac{3}{4}$ or $\frac{1}{2}$?” Take a vote on this one then ask, “Are the pieces the same size? Do we have the same number of pieces in each number?” I would probably ask the class how many think this is a harder problem. Refer back to the ft vs. yards problem and ask which units they changed, the larger or the smaller units. Since they almost certainly converted yards to feet, ask, “Which are the bigger pieces in the fractions in this problem?” “How can we change the halves into fourths?” Make the conversion and decide which is larger. Do this for several more examples where one denominator is a factor of the other.

Again, do more examples until the students can convert to common denominators.

At some point it is important to ask, “What needs to be true about the size of the pieces if we want to compare the size of the fractions?” And, “What part of the fraction tells us the size of the pieces?”

D. Denominators with no common factors. [A good warm up problem for this section would be to ask, “ What number is in both the thirds family and the fifths family?”]
Ask, “Which is larger $\frac{2}{3}$ or $\frac{3}{5}$?” After taking a vote, ask, “Can we convert one into the other like in the previous examples?” “Is there something we could convert each one into so we would have a common denominator (I used common denominator here but you might want to just use “same size pieces”)?” If the students don’t know at this point

you can recreate both families until they see 15 works. Again, do a few more examples until the students see they can use the product of the denominators as the common denominator in each case.

Again, more examples . . .

E. Denominators with a common factor. Ask, " Which is larger $\frac{3}{4}$ or $\frac{5}{6}$?" It is fine if the students use 24 as the denominator but after answer the question of which is larger, ask, "Could anyone have done it with a denominator smaller than 24?"

Again, do several more examples. Keep in mind that it is probably more intuitive to use 24 in the example above than to use least common multiple. For a treatment of least common multiple, see Appendix B.

V. Addition

A. Start with adding whole numbers on a number line. For example 2 + 3 means taking a line segment of length 2, extending from 0 to 2 and attaching a line segment of length 3 to the end of the first. Do this for a couple of examples stressing that the numbers represent lengths.

B. Fractions with common denominators. Now ask, "Lets see what would happen if we took a length of $\frac{2}{5}$ and added a length of $\frac{1}{5}$." Have the students construct this on a number line. You can let them try this on their own, get the picture of the number line with fifths marked off first and let them finish or do the whole thing together as a class. The important thing to ask along the way is "Are the pieces the same size?" Then, "So if I have 1 piece and add two more pieces that are the same size, how many pieces do I have now?" Then, "Is the size of the pieces the same as before or different?"

Do this with a few more examples, common denominators, each time asking about the size of the pieces. The big deal is to emphasize the size of the pieces and have the kids tell you that if you put them all together, the size of the pieces does not change.

Ask, "When we added these fractions, why didn't we add the denominators?"

Desired response, "the denominator tells us the size of the pieces and the size didn't change."

C.* Follow the same format but with variables in the denominators. For example, $\frac{2}{a}+\frac{3}{a}$. In each case, ask the same questions as in V. about the size of the pieces. Make the denominators as complicated as you think the students will be able to handle.

D. Fractions where one denominator is a factor of the other. Ask, "Suppose I ask you to add $\frac{1}{2}$ and $\frac{1}{4}$, how is this different from the problems we did before?" [I would have done a warm up problem comparing the size of similar fraction.] "How many agree that the pieces are not the same size?" "Could you do something to one of them to make the pieces the same size?" If this doesn't work, refer back to comparing fractions and ask, "When we compared fractions, which fraction did we change to get the pieces the same size?" Complete the process of converting $\frac{1}{2}$ to $\frac{2}{4}$ and add as before.

Do several more examples of this type again getting the kids to articulate the pieces need to be the same size in order to do the addition. [You definitely want to use the term "common denominator but don't lose the idea that this means same size pieces. This will make it much more clear why you do not add the denominators.]

At some point it is important to ask, "What needs to be true about the size of the pieces if we want to add fractions?" And, "What part of the fraction tells us the size of the pieces?"

E.* Follow the same path as in D but the algebra question is $\frac{1}{x}+\frac{2}{x}$. Again, look for problems that gradually get a bit more difficult but with the same condition; one denominator is a factor of the other. For example, $\frac{3}{x+2}+\frac{5}{3x+6}$ The only limitation is their ability to factor.

F. Fractions where the denominators are relatively prime (no common factors). Ask the students to add $\frac{1}{2}$ and $\frac{1}{3}$. Let them play with this a bit to see that this is not the same as the previous case. [It may be necessary to ask, "Are the pieces the same size?" "Can we do something to one of them to make the pieces the same size?" We would like them to see that they can cut the halves into 3 equal pieces and the thirds into 2 equal pieces.

Again, more examples so that they eventually see that they can always multiply the denominators to get the pieces the same size.

G.* Same as in F but this time an example like $\frac{3}{n}+\frac{5}{m}$.

H. Using Least Common Multiple to do the same as in F. [See appendix B for a treatment of LCM.]

VI. Subtracting Fractions.

Follow the progression on addition starting with subtraction of whole numbers (where the answer is positive). This is, 6 – 4 is taking a length of 6 and taking away a length of 4.

VII. Multiplication

For the purpose of this section, $3 \cdot 5$ means 3 groups of 5 or $5 \cdot 5 \cdot 5$. It is also assumes that the students will be comfortable with problems like $\frac{1}{3}$ of 27. [This would have been done previously by asking this type of question in head problems or mental arithmetic.]

A. A fraction times a whole number where the answer is a whole number.

This is just an extension of the head problem question above. When doing a head problem, just mental arithmetic, ask $\frac{1}{3}$ of 27 when doing the problem and change the language to $\frac{1}{3}$ times 27 when checking. Do enough examples of this and then have the students explain how they are getting their answers. This should get them to see that multiplying by $\frac{1}{3}$ is the same as dividing by 3.

Now switch the question to $\frac{2}{3}$ times 27 by asking, "If $\frac{1}{3}$ of 27 is 9, what is $\frac{2}{3}$ of 27?" Repeat this with several examples eliminating the "if $\frac{1}{3}$ of 27 is 9" part of the questions when you think the students will do that part on their own. The idea is to get them to see that multiplying by say $\frac{4}{5}$ means you first take $\frac{1}{5}$ of the number then take 4 of those.

B. A whole number times a fraction when the answer is not a whole number.

Do an example of multiplication as repeated addition. For example, ask, " Who can tell me what multiplication problem is represented by 5 + 5 + 5?" Then go the other way, "so what addition problem does $5 \cdot 4$ represent?" [Order does not really matter here but we chose the order mentioned in the note above.]

Now ask, "What addition problem is represented by $4 \cdot \frac{2}{5}$?" Write this out so they see that this is 4 groups of 2 fifths which is 8 fifths. [You may need/want to go back to the denominator determines the size of the pieces and the numerator the number of pieces to focus on the idea that it is the number of pieces that is changing.]

Do several more of these until the students see that one can multiply the whole number by the numerator to get the answer.

Ask, "Who can tell me, without writing out the whole problem, why we can just multiply the numerator by the whole number and leave the denominator the same?"

Desired response, "The whole number tells us how many of the fractions are in the sum and the numerators are all the same, so the numerator is the whole number times the numerator."

C. A fraction times a whole number when the answer is not a whole number.

Reversing the order it is probably good to stick with a unit fraction in the first example. Ask, "What is $\frac{1}{3}$ times 5 or $\frac{1}{3}$ of 5?" [This is done on the number line.] There are two options here. First, it would probably be good to let the students guess an answer and use the idea that this means what gives you five when you add 3 of them to check their answers. If they don't come up with an answer (or you could go here first), ask,

"What is 1/3 of 1?" Then, "If 1/3 of 1 is 1/3, then what is 1/3 of 5?" Done on a number line, the students would shade one third of each unit length. There would then be 5 of those thirds shaded. [This will be an important idea later so be sure they can follow this logic on more examples.]

Ask, "Who can explain why the picture tells us to multiply the whole number by the numerator?"

Desired response, "The numerator tells us that one piece is shaded in each whole and the whole number tells how may wholes there are."

Come back to check on the commutative law by asking, "How many think we should get the same answer we got when we did $5 \cdot \frac{1}{3}$?" and check the answer the same way.

Repeat this with examples using other than unit fractions until the students see that they just need to multiply the whole number by the numerator. That is multiplying by $\frac{4}{5}$ is the same as taking $\frac{1}{5}$ of the number (be prepared to take this as $\frac{1}{5}$ of one and then times the numerator if necessary) and then multiplying by 4. [After a couple of examples, start asking the students if anyone sees a quicker way to do this letting someone tell you when it looks like most have the idea.]

D. A unit fraction times a unit fraction. [What may be a more practical approach to a fraction times a fraction follows in Section VIII. It is fine to skip from here to that section.]]

Again on a number line, ask, "what do you think $\frac{1}{3} \cdot \frac{1}{5}$ equals given that this is the same as $\frac{1}{3}$ of $\frac{1}{5}$?" The students may play with this to come up with an answer. If they don't come up with this you can ask the following sequence of questions.

“Who can tell me how to show where $\frac{1}{5}$ is on the number line?’

“Multiplying by $\frac{1}{3}$ means we need to cut this up into how may pieces?”

“ How many of the new pieces should I shade?”

“How many of these pieces are in the whole?” On this question it may be necessary to ask about “equal size pieces as they probably only cut one of the fifths into 3 pieces.

Again, more examples until the students see a pattern.

Ask, “Who can explain how the number line tells us you multiply the denominators?

Desired response, “ The denominator of one fraction tells how many pieces are in the whole and then we cut one of those into the number of pieces the other fraction tells us to use. You multiply to see how many pieces are now in the whole.”

E. Multiplying a unit fraction times any fraction.

“How many people have an idea what $\frac{1}{3}\cdot\frac{4}{5}=$? [It might be good to talk about this as $\frac{1}{3}$ of $\frac{4}{5}$.] Let them play with this and if they don’t come up with an answer, ask, “What is $\frac{1}{3}$ of $\frac{1}{5}$?” (emphasize the 1) Then, “If $\frac{1}{3}$ of $\frac{1}{5}$ is $\frac{1}{15}$, what would $\frac{1}{3}$ of 4 of those fifths equal?” As usual, do a few of these so the students see you can just multiply the denominators.

F. Multiplying fractions in general.

Refer back to VII. C. 8 above so they remember multiplying by $\frac{4}{5}$ is the same as taking $\frac{1}{5}$ of the number and then multiplying by 4. [Get the students to articulate this by, if necessary, asking, “So when you multiply $\frac{4}{5}\cdot 20$, what is the first step?” Then, “What is the second step?”]

Now, “ if you multiply $\frac{4}{5}$ by $\frac{2}{3}$, what would you do first?” Get the answer to $\frac{1}{5}\cdot\frac{2}{3}$ then ask, “and then what you do to $\frac{2}{15}$?” As usual, do this with several examples until they see the general method.

VIII. Multiplication using the Array Model

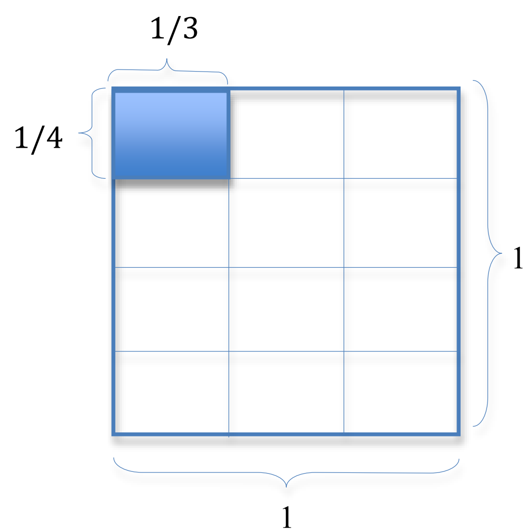
Note: In the previous development of multiplication, one of the numbers represented a distance on the number line and the other number was "unitless." For example, when we did $4 \cdot \frac{1}{3}$, the $\frac{1}{3}$ could represent one-third of a foot and by multiplying by 4, we are taking 4 of those lengths. The result is then 4/3 ft. In the following, both fractions will represent a distance and the product will represent the area of a rectangle.

A. A unit fraction times a unit fraction.

Start by asking the students the area of a few rectangles where the lengths of the sides are whole numbers. Then ask, "Who can draw a picture of a rectangle whose area is found by multiplying $\frac{1}{3}$ and $\frac{1}{4}$?" Get a picture on the board and see if anyone has a guess for the area.

Now draw a 1/3 by 1/4 rectangle and a 1 by 1 rectangle on the board and ask, "Can anyone fit the this rectangle (pointing to the 1/3 by 1/4 rectangle inside the unit square?" Then, "What can we add to the picture to see how many of these smaller rectangles fit in the unit square?"

The goal is to get the picture shown on the right, on the board.



Now ask,

"Are the pieces the same size?"

"How many pieces are shaded?" (Write the 1 in the numerator after Area $= \frac{1}{3} \cdot \frac{1}{4} =$)

"How many pieces are in the whole?" (Write the 12 in the denominator)

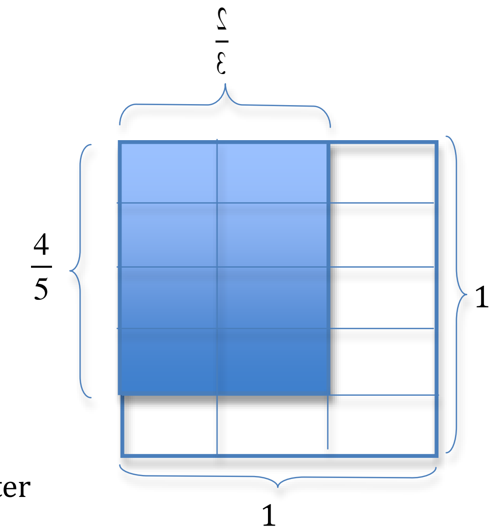
"So, $\frac{1}{3} \cdot \frac{1}{4} =$?"

Ask, "How many people got 12 by counting the rectangles?" And then, "How many got it without counting?" and ask them to explain how they did it without counting.
Do a few more examples like this, each time asking them to make a prediction for what the answer is going to be, again, following the usual pattern of questioning to get them to generalize.

Ask, "How do you know from the picture that you always multiply the denominators?

Desired Response, "The denominators tell us how many rectangles are across the top and how many are down the side. So, you don't have to count the rectangles, just multiply."

B. A fraction times a fraction.



Write the problem $\frac{2}{3}\cdot\frac{4}{5}=$ on the board.
Follow the same set of questions we used
in a unit fraction times unit fraction above to get the picture shown
on the right on the board.

Now ask,
"Are the pieces the same size?"

"How many pieces are shaded?" (Write 8 in the numerator after Area = $\frac{2}{3}\cdot\frac{4}{5}=$)
"How many pieces are in the whole?" (Write the 15 in the denominator)
"So, $\frac{2}{3}\cdot\frac{4}{5}=$?"

Ask, "How many people got 15 by counting the rectangles?" And then, "How many got it without counting?" Ask them to explain how they did it without counting.
Do a few more examples like this, each time asking them to make a prediction for what the answer is going to be, again, following the usual pattern of questioning to get them see that the number of pieces is the product of the numerators and the number of pieces in the whole is the product of the denominators.

Ask, "We know from before why we multiplied the denominators but why do we multiply the numerators?"

Desired Response, " Just like with the denominators, the numerators tell you how many rectangles are shaded across the top and down the side. So, you don't need to count the shaded rectangles, just multiply."

IX. Division

A. A fraction divided by a whole number.

Ask, "How many people already know how to divide a whole number by a whole number, like 20 divided by 4?" "I wonder what would happen if we divided a fraction by a whole number."
Ask, "How many have an idea what $\frac{1}{3}\div 5$ equals?" Let the students play with bit. (We are assuming that they will divide the pieces into 5 pieces and get 15 pieces in the whole. Do a couple of more examples with unit fractions then switch to general fractions.

Ask, "If 1/3 divided by 5 equals 1/15, what do you think 2/3 divided by 5 equals?" [The idea here is to get the students to realize they can cut each of the thirds into 5 equal pieces and give each person 2 of them.] You could ask, "How many agree that $\frac{2}{3} = 2 \cdot \frac{1}{3}$?" So, "how many agree that $\frac{2}{3} \div 5 = 2 \cdot \frac{1}{3} \div 5$?" and get that this is the same as $2 \cdot \frac{1}{15}$.

B. Dividing a whole number by a unit fraction.

Ask, "How do we determine how many 4's are in 20?" Do this on the number line and connect this to a division problem. Make sure the students know this is $20 \div 4$.
Now ask, "How many 1/3's are in 5?"
"What division problem answers this question?" After getting $5 \div \frac{1}{3}$ on the board, send them to the number line to find the answer. If they get an answer, ask how they got it. In any case, it would be good to ask, "How many 1/3's are in 1?" And then, "So how many 1/3's are in 5?" [This is a good chance to review the number line and mixed numbers.] Do this with several more examples getting the students to see this is the same as 3 times 5 as there are 3 1/3's in 1 so there must be $3 \cdot 5$ of them in 5.

Ask, " What question do you ask yourself when you do 18 divided by 3?" If you don't get an answer ask, "How many people think about this as what do you multiply 3 by to get 18?" Now write on the board

$18 \div 3 = ? \rightarrow 3 \cdot ? = 18$ Get several of these on the board, one under the other, until the students have the pattern.

Say, "Let's go back and look at what we just did another way."
Now, going back to the original problem, ask "so $5 \div \frac{1}{3}$ =? is the same as what multiplication problem?"

Now we have $\frac{1}{3} \cdot \square = 5$. Ask, "What did we multiply 1/3 by to get 1?" Put 3 in the box [$\frac{1}{3} \cdot \boxed{3} = 5$] and ask, " So 1/3 times 3 is _____?" "Now what do we have to multiply that by to get 5?" Now you have $\frac{1}{3} \cdot \boxed{3 \cdot 5} = 5$ in the box so you can ask, pointing to the $3 \cdot 5$, "So what did we multiply 1/3 by to get 5?"

Do a few more examples using the same process. In each case have the students make a prediction for what the answer is going to be.

B. Dividing a fraction by a fraction.

Make sure the students know that 3/2 is the number you would multiply 2/3 by to get 1.

Ask, "What would you get if you divided $\frac{2}{5}$ by $\frac{3}{4}$?" Let the students play with this a bit not expecting an answer**.** "So let's go back and look at this the way we did the last problems."

Now ask them to rewrite $\frac{2}{5} \div \frac{3}{4} = ?$ as a multiplication problem. Now we have $\frac{3}{4} \cdot \square = \frac{2}{5}$ on the board, replacing the ? with a box. [Make sure you have some of the problems from section B still on the board.] Refer back to one of the problems from section B and ask why we picked the first number in the box. [If you don't get an answer, you can ask, "What did we always get when we multiplied the first number in the box by the number outside the box?"

Now ask, "So what do we multiply 3/4 by to get 1?" You have $\frac{3}{4} \cdot \boxed{\frac{4}{3} \quad\quad} = \frac{2}{5}$ on the board.

"And what do we multiply 1 by to get 2/5?" You now have $\frac{3}{4} \cdot \boxed{\frac{4}{\ } \cdot \frac{2}{\ }} = \frac{2}{5}$ on the board.

"So what did we multiply 3/4 by to get 2/5?" Again, pointing to the numbers in the box.

Repeat this a few more times each time asking if anyone has a prediction what the answer will be before you do the process. The idea is to get the students to see that the "missing number" is the reciprocal of the number you are dividing by multiplied by the number you are dividing into.

Appendix A: Comparing Lengths of Different Units

I. Comparing like units:

Ask the students, "Which is longer 4 feet or 3 feet?" Do a couple of more examples using yards, hours, minutes and any other units with which the students would likely be comfortable.

II. Comparing when one unit is a "subunit" of the other.

Ask, "Which is longer, 2 weeks or 8 days?" [Avoid asking how many days or weeks this would be to see how the students deal with the different units.] Ask them to explain how they got their answer.

Repeat this asking them to compare, for example, in each case ask them to explain how they got their answer.

3 feet and 5 inches

200 seconds and 2 minutes

4 yards and 5 feet

After doing several of these, ask, "How many people notice something similar about what you did in each problem?" [The goal here is to get them to see that in each case, they probably converted the larger units into the smaller units.] If you don't get an answer, simply ask if they converted the larger units to the smaller units or the other way around for each example. Once they get this you might give them another to make sure they can follow that strategy again.

III. The smaller unit is not a subunit of the larger.

Ask them how comparing 2 dimes and 3 quarters is different from the examples above. [If you don't get an answer, ask, "Can you convert dimes to quarters? How about quarters to dimes?"] Once they agree one can't be converted directly to the other, ask them to explain how they would add these two amounts of money.

Once they see it is necessary to convert both to some other common unit, you can try another example say 100 yards and 1 mile. [I would only do this if I thought the students could handle the conversions and the size of the numbers. If not, it will not add to the idea that both units need to be converted. If so, I would let one example suffice.]

Appendix B: Least Common Multiple (Denominator)

Before doing this, students should be able to find greatest common factors (gcf). We will also be referring back to the families of fractions discussed in III. F. and comparing fractions.

Ask the students, "When we compared fractions with denominators 3 and 5, what was the common denominator we used?" Do a few of these so that they recall $a \cdot b$ will always be in the family starting with 1/a and the family starting with 1/b.

Now ask, "So what would you expect to be in the families starting with 1/6 and 1/4?" Assuming they say 24, ask, "Is there a smaller number that is in both families?" Once they see that 12 is in both families, ask, "12 is 24 divided by what?" Record the answers to each of these questions on the board.

Do a few more of these where the numbers have a common factor, in each case asking the same questions as above. After a few, if a none of the students has recognized that the answer to the last question ("12 is 24 divided by what in the case above"), ask them the greatest common factor of the two denominators in each case. Once they see this, you can ask, " Does anyone see a way to get the least common denominator from the product and the greatest common factor?" The goal is to get them to see that the

Least Common Denominator for $\frac{1}{a}$ and $\frac{1}{b}$ = $\frac{a \cdot b}{\text{greatest common factor}}$